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ON THE KOLOSOV- MUSKHELISHVILI ANALOG FOR THE THREE-DIMENSIONAL STATE OF STRESS*

O.G. GOMAN

The axisymmetric problem of the theory of elasticity includes the known representation of the stresses and displacements in terms of two p-analytic functions /1/, which is an analog of the Kolosov- Muskhelishvili formulas for the plane problem. Below analogous formulas are derived for every term of the Fourier series which are assumed to represent over the angular coordinate the stresses and displacements in the non-axisymmetric stress state.

1. Consider the differential operators

$$\bar{M}_k = \begin{Vmatrix} r^k \dfrac{\partial}{\partial r} & -\dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial z} & \dfrac{1}{r^k}\dfrac{\partial}{\partial r} \end{Vmatrix}, \quad K_k = \begin{Vmatrix} \dfrac{\partial}{\partial r} & -\dfrac{\partial}{\partial z} r^k \\ \dfrac{\partial}{\partial z} & \dfrac{1}{r^k}\dfrac{\partial}{\partial r} \end{Vmatrix}$$

$$\bar{K}_k = \begin{Vmatrix} r^k \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial z} \\ -\dfrac{\partial}{\partial z} & \dfrac{1}{r^k}\dfrac{\partial}{\partial r} \end{Vmatrix}, \quad M_k = \begin{Vmatrix} \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial z} r^k \\ -\dfrac{\partial}{\partial z} & \dfrac{1}{r^k}\dfrac{\partial}{\partial r} \end{Vmatrix}$$

In accordance with the terminology adopted by G.N. Polozhii in /1/, we shall call the function

$$f(z, r) = p(z, r) + iq(z, r) \equiv \binom{p}{q}$$

r^k-analytic (or simply (k)-analytic), $(-k)$ analytic, (k)-antianalytic and $(-k)$ antianalytic, if

$$\bar{M}_k \binom{p}{q} = 0, \quad K_k \binom{p}{q} = 0, \quad \bar{K}_k \binom{p}{q} = 0, \quad M_k \binom{p}{q} = 0 \tag{1.1}$$

respectively. The operator $\bar{M}_k$ is an analog of the operator of differentiation with respect to the conjugate variable $\partial/\partial\bar{\zeta}$ in the complex plane $\zeta = r + iz$ (at $k = 0$ the above operators become identities). We note that for any (k)-analytic function $f = p + iq$ there exists a (k)-analytic "primitive" function $F = P + iQ$, such that $f = \partial F/\partial z$. The following relations hold for the (k)-analytic function $P + iQ$:

$$\bar{M}_k \binom{P}{-Q} = 2\binom{\partial Q/\partial z}{\partial P/\partial z}, \quad M_k \binom{\partial Q/\partial z}{\partial P/\partial z} = 0 \tag{1.2}$$

$$\bar{M}_k \binom{z\partial P/\partial z}{z\partial Q/\partial z} = \binom{-\partial Q/\partial z}{\partial P/\partial z}$$

2. We write the equations of the theory of elasticity as follows /2/:

$$\nabla^2 w + \frac{1}{1-2\nu}\frac{\partial\vartheta}{\partial z} = 0, \quad \vartheta = \frac{\partial w}{\partial z} + \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta}$$

$$\left(\nabla^2 - \frac{1}{r^2}\right) u - \frac{2}{r^2}\frac{\partial v}{\partial\theta} + \frac{1}{1-2\nu}\frac{\partial\vartheta}{\partial r} = 0$$

$$\left(\nabla^2 - \frac{1}{r^2}\right) v + \frac{2}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{1-2\nu}\frac{1}{r}\frac{\partial\vartheta}{\partial\theta} = 0$$

where z, r, θ is the cylindrical coordinate system, w denotes the axial, u the radial and v the tangential displacements. Assuming that the displacements can be represented by the Fourier series

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$$w = \omega_0^1 + \sum_{n=1}^{\infty} (w_n^1 \cos n\theta + w_n^2 \sin n\theta)$$
$$u = u_0^1 + \sum_{n=1}^{\infty} (u_n^1 \cos n\theta + u_n^2 \sin n\theta)$$
$$v = v_0^2 + \sum_{n=1}^{\infty} (-v_n^1 \sin n\theta + v_n^2 \cos n\theta)$$

we obtain the following system for values $w_n^{1,2}$, $u_n^{1,2}$, $v_n^{1,2}$, $n \geqslant 1$ (from now on the superscripts will be omitted):

$$\Delta_n w_n + \frac{1}{1-2\nu} \frac{\partial \vartheta_n}{\partial z} = 0 \tag{2.1}$$
$$\left(\Delta_n - \frac{1}{r^2}\right) u_n + \frac{2n}{r^2} v_n + \frac{1}{1-2\nu} \frac{\partial \vartheta_n}{\partial r} = 0 \tag{2.2}$$
$$\left(\Delta_n - \frac{1}{r^2}\right) v_n + \frac{2n}{r^2} u_n + \frac{1}{1-2\nu} \frac{n}{r} \vartheta_n = 0 \tag{2.3}$$
$$\vartheta_n = \frac{\partial w_n}{\partial z} + \frac{\partial u_n}{\partial r} + \frac{u_n - n v_n}{r} \tag{2.4}$$
$$\left(\Delta_n = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{n^2}{r^2}\right)$$

The stress state symmetrical with respect to the z axis will not be discussed; its representation in terms of the p-analytic functions was given in /1/. Combining the equations (2.2) and (2.3) we obtain

$$\Delta_{n+1}(u_n - v_n) + \frac{1}{1-2\nu}\left(\frac{\partial \vartheta_n}{\partial r} - \frac{n}{r}\vartheta_n\right) = 0 \tag{2.5}$$
$$\Delta_{n-1}(u_n + v_n) + \frac{1}{1-2\nu}\left(\frac{\partial \vartheta_n}{\partial r} + \frac{n}{r}\vartheta_n\right) = 0 \tag{2.6}$$

and making the substitution

$$w_n = r^n Z_n,\ u_n - v_n = r^{-(n+1)} Y_n,\ \vartheta_n = r^n \Theta_n$$

we reduce (2.1) and (2.5) to

$$M_{2n+1}\bar{M}_{2n+1}\begin{pmatrix} Z_n \\ Y_n \end{pmatrix} = -\frac{1}{1-2\nu}\begin{pmatrix} r^{2n+1} & \partial\Theta_n/\partial z \\ & \partial\Theta_n/\partial r \end{pmatrix} \tag{2.7}$$

which can be splitted into two first order equations

$$M_{2n+1}\begin{pmatrix} s_n \\ t_n \end{pmatrix} = -\frac{1}{1-2\nu}\begin{pmatrix} r^{2n+1} & \partial\Theta_n/\partial z \\ & \partial\Theta_n/\partial r \end{pmatrix} \tag{2.8}$$
$$\bar{M}_{2n+1}\begin{pmatrix} Z_n \\ Y_n \end{pmatrix} = \begin{pmatrix} s_n \\ t_n \end{pmatrix} \tag{2.9}$$

solvable one after the other.

We note that Θ_n is a $(2n+1)$-harmonic function (since ϑ is harmonic). The solution of (2.8) can be expressed, according to (1.1), by two $(2n+1)$-analytic functions of the form

$$\begin{pmatrix} s_n \\ t_n \end{pmatrix} = \begin{pmatrix} h_n \\ f_n \end{pmatrix} + \begin{pmatrix} q_n \\ -p_n \end{pmatrix}$$

where the first term represents the solution of the homogeneous equation and the second term a particular solution of the inhomogeneous equation. Moreover, from (2.8) we see that

$$p_n = \frac{1}{2(1-2\nu)}\Theta_n \tag{2.10}$$

Now the lower equation of the system (2.9) leads to

$$f_n = \varkappa p_n + a_n;\ \varkappa = 3 - 4\nu$$
$$a_n = r^{-n}\left(\frac{n u_n}{r} - \frac{1}{r}\frac{\partial}{\partial r}(r v_n)\right)$$

We write the function h_n conjugate to f_n in the analogous form

$$h_n = \varkappa q_n + b_n$$

where $a_n + ib_n$ is an arbitrary (for the time being) $(2n+1)$ analytic function. Bringing in the "primitives" $P_n + iQ_n$ and $A_n + iB_n$ for $p_n + iq_n$ and $a_n + ib_n$, we write (2.9) in the form

$$\bar{M}_{2n+1}\begin{pmatrix} Z_n \\ Y_n \end{pmatrix} = \varkappa\begin{pmatrix} \partial Q_n/\partial z \\ \partial P_n/\partial z \end{pmatrix} + \begin{pmatrix} \partial Q_n/\partial z \\ -\partial P_n/\partial z \end{pmatrix} + \begin{pmatrix} \partial B_n/\partial z \\ \partial A_n/\partial z \end{pmatrix}$$

The general solution of the above equation based on the properties of (2.1), is

$$\begin{pmatrix} Z_n \\ Y_n \end{pmatrix} = \frac{\varkappa}{2}\begin{pmatrix} P_n \\ -Q_n \end{pmatrix} - \begin{pmatrix} z\partial P_n/\partial z \\ z\partial Q_n/\partial z \end{pmatrix} + \frac{1}{2}\begin{pmatrix} A_n \\ -B_n \end{pmatrix} + \begin{pmatrix} \Phi_n \\ \Psi_n \end{pmatrix} \tag{2.11}$$

where $\Phi_n + i\Psi_n$ is an arbitrary $(2n+1)$ analytic function.

When $n = 0$, (2.11) yields the representation of G.N. Polozhii for the axisymmetric problem, which was obtained by a more complicated method (involving the techniques of p-integration). Introducing the substitutions

$$w_n = r^{-n}Z_n', \quad u_n + v_n = r^{n-1}X_n', \quad \vartheta_n = r^{-n}\Theta_n'$$

we can write (2.6) and (2.1) in the form

$$K_{2n-1}\bar{K}_{2n-1}\begin{pmatrix} X_n' \\ Z_n' \end{pmatrix} = -\frac{1}{1-2\nu}\begin{pmatrix} \partial\Theta_n'/\partial r \\ r^{-2n+1}\partial\Theta_n'/\partial z \end{pmatrix}$$

Solving this equation analogously to the previous one, we obtain

$$\begin{pmatrix} X_n' \\ Z_n' \end{pmatrix} = -\frac{\varkappa}{2}\begin{pmatrix} P_n' \\ Q_n' \end{pmatrix} - \begin{pmatrix} z\partial P_n'/\partial z \\ -z\partial Q_n'/\partial z \end{pmatrix} - \frac{1}{2}\begin{pmatrix} A_n' \\ B_n' \end{pmatrix} + \begin{pmatrix} \Phi_n' \\ \Psi_n' \end{pmatrix} \tag{2.12}$$

where $P_n' + iQ_n'$, $A_n' + iB_n'$ and $\Phi_n' + i\Psi_n'$ are arbitrary $(2n-1)$-analytic functions and

$$\frac{\partial Q_n'}{\partial z} = -\frac{1}{2(1-2\nu)}\Theta_n', \quad \frac{\partial B_n'}{\partial z} = r^{2n}\frac{\partial A_n}{\partial z} \tag{2.13}$$

The representations (2.11) and (2.12) contain six arbitrary functions. However, only three of these functions are independent by virtue of the relations

$$Q_n' = -r^{2n}P_n, \; B_n' = r^{2n}A_n$$

which follow from (2.10) and (2.13), and also of the equation $r^nZ_n = r^{-n}Z_n'$ which yields

$$A_n + \Phi_n + r^{-2n}\Psi_n' = 0$$

Thus the expressions (2.11) and (2.12) contain only three basic arbitrary functions, and we can choose, as these functions, e.g. $P_n + iQ_n$, $\Phi_n + i\Psi_n$ and $\Phi_n' + i\Psi_n'$. The formulas (2.11) and (2.12) can be written in a different form by expressing the functions A_n' and B_n in terms of the basic functions. Using the condition of the $(2n+1)$-analyticity of (1.1), we find that

$$B_n = -\Psi_n + r^{2n}\Phi_n' + 2n\psi_n'$$

where ψ_n' is a $(-2n+1)$- harmonic function such that

$$\Psi_n' = \frac{\partial\psi_n'}{\partial z}, \quad \Phi_n' = -\frac{1}{r^{2n-1}}\frac{\partial\psi_n'}{\partial r}$$

In the same manner we find for A_n'

$$A_n' = -\Phi_n' + r^{-2n}\Psi_n + 2n\varphi_n$$

where the $(2n+1)$-harmonic function φ_n is such that

$$\Phi_n = \frac{\partial\varphi_n}{\partial z}, \quad \Psi_n = r^{2n+1}\frac{\partial\varphi_n}{\partial r}$$

The representations (2.11) and (2.12) can now be written as

$$2w_nr^{-n} = \varkappa P_n - 2z\frac{\partial P_n}{\partial z} + \frac{\partial\varphi_n}{\partial z} - r^{-2n}\frac{\partial\psi_n'}{\partial z} \tag{2.14}$$

$$2(u_n - v_n)r^{n+1} = -\varkappa Q_n - 2z\frac{\partial Q_n}{\partial z} + 3r^{2n+1}\frac{\partial\varphi_n}{\partial r} + r^{2n+1}\frac{\partial}{\partial r}(r^{-2n}\psi_n')$$

$$2(u_n + v_n)r^{-n+1} = -\varkappa P_n' - 2z\frac{\partial P_n'}{\partial z} - 3r^{-2n+1}\frac{\partial\psi_n'}{\partial r} - r^{-2n+1}\frac{\partial}{\partial r}(r^{2n}\varphi_n)$$

Here the functions Q_n and P_n' are expressed in terms of P_n, since Q_n is a conjugate of P_n and $Q_n' = r^{-2n}P_n$ is a conjugate of P_n'

The formulas (2.11) and (2.12) or (2.14) are three-dimensional analog of the Kolosov-Muskhelishvili formulas. They contain three arbitrary real functions (e.g. φ_n, P_n, ψ_n') so that according to the representation obtained every group of the coefficients of the Fourier series of the same name (w_n^1, u_n^1, v_n^1) and (w_n^2, u_n^2, v_n^2) is expressed only in terms of three arbitrary functions of the type shown. We shall not concern ourselves with the representation of the stresses.

Let us compare the representation obtained with that of P.F. Papkovich, which has the following form in the cylindrical coordinate system:

$$w = \frac{\partial}{\partial z}(\varphi + r\psi_r + z\psi_z) - 4(1-\nu)\psi_z \tag{2.15}$$
$$u = \frac{\partial}{\partial r}(\varphi + r\psi_r + z\psi_z) - 4(1-\nu)\psi_z$$
$$v = \frac{1}{r}\frac{\partial}{\partial \theta}(\varphi + r\psi_r + z\psi_z) - 4(1-\nu)\psi_\theta$$

where φ is a harmonic function and $\psi(\psi_z, \psi_r, \psi_\theta)$ a harmonic vector. If we expand all the functions into Fourier series in θ, then the group (w_n^1, u_n^1, v_n^1) of the coefficients will be expressed in terms of four arbitrary functions $(\varphi_n^1, \psi_{zn}^1, \psi_{rn}^1, \psi_{\theta n}^1)$ and the group (w_n^2, u_n^2, v_n^2) of the coefficients by four arbitrary functions $(\varphi_n^2, \psi_{zn}^2, \psi_{rn}^2, \psi_{\theta n}^2)$ where $\varphi_n^1, \varphi_n^2, \psi_{zn}^1, \psi_{zn}^2 \ldots$ are the coefficients of the Fourier series for φ, ψ_z, ψ_r and ψ_θ. This implies that for every group of the Fourier series coefficients of the same kind the number of arbitrary functions in the representation obtained per unit is smaller than that in (2.15), and for this reason we can speak of advantage of the representation obtained here over that of Papkovich (and certain other representations).

The formulas (2.14) become most compact when three $(2n+1)$-harmonic functions ω_n, μ_n, ν_n, are chosen as the basic functions and are such that

$$\mu_n = -2\varphi_n - \psi_n, \quad \nu_n = -2\psi_n$$
$$P_n = \frac{\partial \omega_n}{\partial z}, \qquad Q_n = r^{2n+1}\frac{\partial \omega_n}{\partial r}, \qquad P_n' = r\frac{\partial \omega_n}{\partial r} + 2n\omega_n$$
$$Q_n' = -r^{2n}\frac{\partial \omega_n}{\partial z}$$
$$A_n = \frac{\partial \psi_n}{\partial z}, \qquad B_n = r^{2n+1}\frac{\partial \psi_n}{\partial r}$$
$$A_n' = -r\frac{\partial \psi_n}{\partial r} - 2n\psi_n, \qquad B_n' = r^{2n}\frac{\partial \psi_n}{\partial z}$$
$$\Phi_n = \frac{\partial \varphi_n}{\partial z}, \qquad \Psi_n = r^{2n+1}\frac{\partial \varphi_n}{\partial r}$$
$$\Phi_n' = r\left(\frac{\partial \varphi_n}{\partial r} + \frac{\partial \psi_n}{\partial r}\right) + 2n(\varphi_n + \psi_n)$$
$$\Psi_n' = -r^{2n}\left(\frac{\partial \varphi_n}{\partial z} + \frac{\partial \psi_n}{\partial z}\right)$$

In place of (2.14) we now obtain

$$2w_n = r^n\left(\varkappa\frac{\partial \omega_n}{\partial z} - 2z\frac{\partial^2 \omega_n}{\partial z^2} - \mu_n\right)$$
$$2(u_n - v_n) = -r^{-n}\frac{\partial}{\partial r}\left(\varkappa\omega_n + 2z\frac{\partial \omega_n}{\partial r} + \mu_n - \nu_n\right)$$
$$2(u_n + v_n) = -r^{-n}\frac{\partial}{\partial r}r^{2n}\left(\varkappa\omega_n + 2z\frac{\partial \omega_n}{\partial z} + \mu_n + \nu_n\right)$$

Expressing in the above formulas every one of the r^{2n+1}-harmonic functions ω_n, μ_n and ν_n, with help of the integral Polozhii operator, by an analytic function /1/, we obtain the representation for the three-dimensional displacements (w_n, u_n, v_n) in terms of three arbitrary analytic functions. This representation coincides exactly with that of A.Ia. Aleksandrov-Iu. I. Solov'eva /12/ and can therefore serve to provide a stronger justification for the somewhat artificial method of representing the three-dimensional stress state in terms of the auxilliary two-dimensional state developed by the above authors.

Establishing the admissible arbitrariness of the functions used for the given displacements reduces to the process of determining these functions from, e.g. the expressions (2.14) with the zero left-hand parts. This yields

$$P_n = -\frac{1}{2n} S_n r^{-2n} + K_n, \quad Q_n = S_n z + f_n \tag{2.16}$$
$$P_n' = 2nK_n z + g_n', \quad Q_n' = \frac{1}{2n} S_n - K_n r^{2n}$$
$$\varphi_n = -\frac{1}{8n} [(2S_n z + \varkappa f_n + \alpha_n) r^{-2n} + 4n(1 + 2\varkappa) K_n z + \varkappa g_n' - 6n\beta_n]$$
$$\psi_n' = -\frac{1}{8n} [2(1 + 2\varkappa) S_n z + \varkappa f_n - 3\alpha_n + r^{2n}(4nK_n z + \varkappa g_n' + 2n\beta_n)]$$

where $S_n, K_n, f_n, g_n', \alpha_n$ and β_n are constants. The degree of arbitrariness of the functions $w_n^{1,2}$, $u_n^{1,2}$ and $v_n^{1,2}$ under the given stresses is found in the same manner, and this yields the following result:

a) $n = 1$. The arbitrariness of the functions P_1, Q_1, P_1' and Q_1' is determined by the formulas (2.16); for the functions φ_1 and ψ_1' we must supplement the right-hand sides of the formulas (2.16) with the terms $-2b_1 z - {}^1/_2 d_1$ and $-{}^1/_2 d_1 r^2$, respectively (b_1 and d_1 are constants);

b) $n \geqslant 2$. The degree of arbitrariness is determined by the formulas (2.16).

In conclusion we note that the present approach can also be used for compact derivation of the Kolosov-Muskhelishvili formulas. Thus the proposed method gives a unique procedure for obtaining the Kolosov-Muskhelishvili and Polozhii formulas and their generalizations to the non-axially symmetric case.

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